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Clifford algebra and the structure of point groups in higher-dimensional spaces

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Abstract
With the basic Clifford units being identified as mirrors, it is demonstrated how proper and improper symmetry operations of point groups in spaces of arbitrary dimensions can be parametrized. In such an approach consistency with parametrizations for groups in three dimensions can be achieved even if double groups are considered. The conversion of Clifford parameters into Cartesian matrices and vice versa is discussed and, for rotations in $\mathbb{R}^4$, also the parametrization in terms of pairs of rotations in $\mathbb{R}^3$. The formalism is illustrated by a number of examples.

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1. Introduction

Although point groups in spaces of dimension 4 and larger are fairly well known, a systematic study of their structures has so far been somewhat sketchy, despite the pioneering work of Goursat [1], Hurley [2], Mozrzymas and Solecki [3] and various other authors. As an example of puzzling structural problems still remaining in the literature, although Judd and Lister [4] found that the group of the hypercube in four dimensions has formally a double-group structure, this was done merely by consideration of its multiplication table. In fact, no general theory is yet available to justify this structure.

We shall fill in such gaps by studying the general properties of point groups in higher dimensions on using the powerful method of Clifford algebra proposed in our earlier work [5, 6]. We shall provide in this paper a quick revision of the work on Clifford algebra that will be necessary for the applications to point groups to be later considered.

Physical applications of multi-dimensional point-group theory are well known. In modulated crystals a four-dimensional superspace crystal is defined [7–9]. Quasicrystals [10] have quasiperiodicities that can best be studied as projections onto three-space of translational symmetries in higher dimensions [11, 12]. In nuclear physics four-dimensional symmetries have been used by Judd and Lister [4].

2. Basic definitions

Given n elements $e_1, e_2, \ldots, e_n$, with the multiplication rule

$$e_i e_j + e_j e_i = 2\delta_{ij}, \tag{1}$$

then the set

$$1, e_1, \ldots, e_n, e_1e_2, \ldots, e_1e_n, e_2e_3, \ldots, e_2e_n, e_1e_2e_3, \ldots,$$
$$e_1e_2e_3e_4, \ldots, \ldots, \ldots, e_1e_2e_3\cdots e_n, \tag{2}$$

is the Clifford algebra $\mathbb{C}^n$. The *rank* of the terms that appear in (2) is defined as the number of factors in a given term. The results given in this and the following sections are easy to prove and they can be found in the literature [13, 14].

Theorem 1. *The Clifford algebra* $\mathbb{C}^n$ *has* 2^n *elements.*

Theorem 2. *The* 2^n *elements of* $\mathbb{C}^n$ *separate in two sets for which the rank p is even and odd, respectively, and the number of elements in each set is* 2^{n-1}.

3. Nomenclature and conventions

The *dimension* n of the algebra $\mathbb{C}^n$ is the number of elements $e_1, \ldots, e_n$ in it. The *reversion* ρ (if necessary also written as ρ_n) is the single element

$$\rho = e_1e_2\cdots e_n. \tag{3}$$

We shall use the following conventions in this paper:

i, j, k, l	integers;
$m, n, p, \ldots$	dimensions of an algebra;
s, t, u	ranks;
$a, b, c, \ldots$	elements of the algebra, such as $a = e_1e_2$;
a_s	if necessary, element of rank s.

4. Squares of elements

Theorem 3. *Given*

$$a_s = e_1e_2\cdots e_s, \tag{4}$$

then

$$a_s^2 = (-1)^{\frac{s}{2}(s-1)}. \tag{5}$$

Corollary 1

$$a_s^2 = 1, \qquad s = 4l\,(l = 1, 2, \ldots) \quad \textit{or} \quad s = 4l + 1\,(l = 0, 1, \ldots), \tag{6}$$
$$= -1, \qquad \textit{otherwise}. \tag{7}$$

Corollary 2 (reversion)**.**

$$\rho_s^2 = 1, \qquad s = 4l\,(l = 1, 2, \ldots) \quad \textit{or} \quad s = 4l + 1\,(l = 0, 1, \ldots), \tag{8}$$
$$= -1, \qquad \textit{otherwise}. \tag{9}$$

It follows that

$$\rho_s^2 = 1, \qquad n = 1, 4, 5, 8, 9, 12, 13, \ldots \tag{10}$$

$$= -1, \qquad n = 2, 3, 6, 7, 10, 11, \ldots. \tag{11}$$

5. Inverse and conjugation

Given $a_s = e_1 e_2 \cdots e_s$, its inverse a_s^{-1} is such that

$$a_s a_s^{-1} = a_s^{-1} a_s = 1. \tag{12}$$

Clearly,

$$a_s^{-1} = e_s e_{s-1} \cdots e_1. \tag{13}$$

Theorem 4

$$a_s^{-1} = (-1)^{\frac{s}{2}(s-1)} a_s. \tag{14}$$

The conjugate $a_s^{\dagger}$ of a_s by a_t is by definition

$$a_s^{\dagger} = a_t a_s a_t^{-1}. \tag{15}$$

Theorem 5. *If a_t and a_s have l elements in common, then*

$$a_s^{\dagger} = (-1)^{st-l} a_s. \tag{16}$$

Note from (15) that if $a_s^{\dagger}$ equals a_s, then a_s and a_t commute. The necessary and sufficient condition for the commutation of a_s and a_t is therefore, from (16):

$$(-1)^{st-l} = 1. \tag{17}$$

Consideration of the conjugation under the reversion has important consequences for the commutation of the latter. Take

$$a_s^{\dagger} = \rho_n a_s \rho_n^{-1}. \tag{18}$$

In this case, t in (15) is n, the dimension of the algebra, so that we may take $s \leqslant n$ in (18), in which case l (as used in 16) is necessarily s. Therefore, in (17):

$$(-1)^{st-l} = (-1)^{sn-s} = (-1)^{s(n-1)}, \tag{19}$$

from which the following result follows at once.

Theorem 6. *Given an algebra $\mathbb{C}^n$ with terms a_s,*

$$\textit{for } n \text{ odd}, \ \rho_n \textit{ commutes with } \forall\, a_s, \tag{20}$$

$$\textit{for } n \text{ even}, \ \rho_n \textit{ commutes with } a_s, \forall\, s \text{ even}. \tag{21}$$

6. Conjugation and symmetry operations

In order to define a symmetry operation in $\mathbb{R}^n$ we must choose an orthogonal basis in it. A symmetry operation will be a transformation of the basis (active picture) that keeps the metric. Such basis changes can be given by conjugation.

In $\mathbb{R}^3$ we have to span this space by means of three axial vectors **i**, **j**, **k** which, as shown by Altmann [5], can be given by Clifford terms of rank 2:

$$e_3e_2 = \mathbf{i}, \qquad e_1e_3 = \mathbf{j}, \qquad e_2e_1 = \mathbf{k}. \tag{22}$$

A symmetry operation g (which shall also be expressed by Clifford terms of rank 2) acting on one of the vectors **v** of the basis **i**, **j**, **k** is in this procedure expressed as

$$g\mathbf{v} = gvg^{-1}, \tag{23}$$

where v is the Clifford term corresponding to the vector **v**, and the right-hand side of (23) has to be identified with a corresponding axial vector.

Consider for example the transformation under e_3e_2 of the basis vectors in (22). It is easy to prove that

$$e_3e_2\mathbf{i} = \mathbf{i}, \qquad e_3e_2\mathbf{j} = -\mathbf{j}, \qquad e_3e_2\mathbf{k} = -\mathbf{k}. \tag{24}$$

This shows that e_3e_2 is a rotation of $\mathbb{R}^3$ by π around the x axis (which justifies our previous assertion that such operations may be given in terms of Clifford elements).

In $\mathbb{R}^4$ we have six rank-two terms (see (86) of [5] with an error corrected),

$$e_3e_2, e_1e_3, e_2e_1, e_4e_1, e_4e_2, e_4e_3. \tag{25}$$

The conjugates $ge_ie_jg^{-1}$ of these terms under $g = e_3e_2$ are respectively

$$e_3e_2, -e_1e_3, -e_2e_1, e_4e_1, -e_4e_2, -e_4e_3. \tag{26}$$

This shows that this operation g in $\mathbb{R}^4$ may be identified as a rotation by π of $\mathbb{R}^4$ that leaves invariant the hyperline spanned by e_3e_2 and e_4e_1. That is, g is a rotation about this hyperline.

The usual transitive property of conjugation carries on in spaces of higher dimension, which means that two operations that are conjugate to a third one are also conjugate amongst themselves. Thus, the set of all operations conjugate to a given one are conjugate to each other. Such a set is called a *class*.

A general symmetry operation is given by a linear combination of Clifford terms:

$$g = \sum_{\substack{mnpq\\ \alpha\beta\gamma\delta}} a_{mnpq} e_\alpha^m e_\beta^n e_\gamma^p e_\delta^q, \qquad m = 0, 1, \ldots; \alpha, \beta, \gamma, \delta \text{ permutations of } 1, 2, 3, 4. \tag{27}$$

There are important constraints in this expansion owing to the following result.

Theorem 7. *All the Clifford terms in a symmetry operation must have the same rank parity.*

Proof. The parity $\mathcal{P}$ is the operation that changes the sign of all e_i. Clearly, its square is +1 whence its eigenvalues must be ± 1:

$$\mathcal{P}g = \pm g. \tag{28}$$

All the terms of even rank, of course, must have positive parity, while those of odd rank have negative parity. Therefore, for g to satisfy (28) no mixing of terms with different parity is permitted. For the purpose of this theorem the term 1 must be considered of even parity.

From (15) it is clear that conjugation cannot change a term's parity so that all operations in a class must have exactly the same distribution of ranks, all of course of the same parity. □

7. Proper and improper operations

It follows from the above that symmetry operations must belong to either of two types, those of even parity (even rank) and odd parity (odd rank). They are called *proper* and *improper* operations, respectively.

Since on multiplication factors e_i, e_j are at most eliminated in pairs, the following multiplication rules for symmetry operations obtain:

$$\text{proper} \times \text{proper} = \text{proper}; \tag{29}$$

$$\text{proper} \times \text{improper} = \text{improper}; \tag{30}$$

$$\text{improper} \times \text{improper} = \text{proper}. \tag{31}$$

The reversion ρ_n, of course, can be proper or improper, depending on the dimension of the space. From theorem 6 the following important commutation rules obtain for the inversion in relation to the symmetry operations g.

$$\text{If } \rho_n \text{ is } \textit{improper} \text{ it commutes with } \forall g. \tag{32}$$

$$\text{If } \rho_n \text{ is } \textit{proper} \text{ it commutes only with } \forall g \text{ proper}. \tag{33}$$

8. Clifford algebras and groups: gauges

Consider for example the Clifford algebra $\mathbb{C}^3$ which we present in the form given by Altmann [5]:

$$1;\, e_1, e_2, e_3;\, e_3e_2, e_1e_3, e_2e_1;\, e_1e_2e_3. \tag{34}$$

It is important to realize that this set is not a group:

$$e_3e_2e_3e_2 = -1, \tag{35}$$

but we may construct a projective representation

$$e_3e_2e_3e_2 = [e_3e_2, e_3e_2]1. \tag{36}$$

Here the square bracket is a *projective factor* that in comparison with (35) must take the value

$$[e_3e_2, e_3e_2] = -1. \tag{37}$$

A projective factor with the sign thus correctly kept is said to be in the *Cartan gauge* [16]. For some purposes it may be useful to ignore the negative signs that appear in these factors, that is, to adopt the *trivial factor system* in which all projective factors are taken as unity. In this case we say that we operate in the *Pauli gauge*. Unless explicitly stated, all the work in this paper is done in the Cartan gauge.

Equations (15) and (16) for the conjugation (and commutation) can be re-written in terms of projective factors. From these equations,

$$a_t a_s a_t^{-1} = (-1)^{st-l} a_s, \tag{38}$$

$$a_t a_s = (-1)^{st-l} a_s a_t \tag{39}$$

$$= [a_t, a_s] a_s a_t, \tag{40}$$

with

$$[a_t, a_s] = (-1)^{st-l}, \tag{41}$$

where l is the number of common factors of a_s and a_t.

For the reversion, from (18) and (19),

$$\rho_n a_s \rho_n^{-1} = a_s^{\dagger} = (-1)^{s(n-1)} a_s, \tag{42}$$

$$\rho_n a_s = (-1)^{s(n-1)} a_s \rho_n \tag{43}$$

$$= [\rho_n, a_s] a_s \rho_n, \tag{44}$$

with

$$[\rho_n, a_s] = (-1)^{s(n-1)}. \tag{45}$$

Note that in the Pauli gauge the set (34) becomes a group and that all terms, including the reversion, commute.

9. Structure of point groups in $\mathbb{R}^n$

We shall use the following notation.

K an improper group, with proper operations g and improper operations k.

G a group of proper operations g.

Theorem 8. *The proper operations $g \in K$ form an invariant subgroup $G \subset K$, g being a group of proper operations only.*

Proof. First, from (29), the set $\{g\}$ closes, although in the Clifford realization the Pauli gauge might have to be used (but see later, section 9). Secondly,

$$kgk^{-1} \in G, \tag{46}$$

since the rank of kgk^{-1} is twice the rank of k plus the rank of g minus a multiple of 2 (from possibly eliminated factors). Therefore, this rank is even, and G is an invariant subgroup of K. □

Theorem 9. *$G \subset K$ is an invariant subgroup of K of index 2.*

Proof. Write the factor group of K by G:

$$K = G \oplus Gk_1 \oplus Gk_2 \oplus \cdots. \tag{47}$$

From (31),

$$k_2 k_1^{-1} = g_j \in G, \tag{48}$$

$$k_2 = g_j k_1. \tag{49}$$

$$Gk_2 = Gg_j k_1 = Gk_1, \tag{50}$$

which shows that we only have two coset representatives in (47), say 1 and k. Therefore, all improper groups that contain proper operations must be of the form

$$K = G \oplus Gk, \qquad G \subset K, \quad \text{index 2}. \tag{51}$$

□

Theorem 10. *$\rho_n \in K$, n odd. Given*

$$G \in K, \quad \mathbb{R}^n \text{ with } n = 4r + 1, \quad r = 0, 1, \ldots \tag{52}$$

then

$$K = G \otimes \{E \oplus \rho_n\}. \tag{53}$$

Proof. ρ_n is improper so that it does not belong to G and it can be used as k in (51). Also, from (8) with the conditions for n stated there, ρ_n^2 equals the identity, so that $\{E \oplus \rho_n\}$ is a group which from (20) commutes with all $g \in G$. Therefore, (53) follows from (51). Note that the reversion is here an *explicit* operation, as it appears in the product with $E \in G$. □

Theorem 11. *$\rho_n \in K$, n even. Given*

$$G \in K, \quad \mathbb{R}^n \text{ with } n = 4r + 2, \quad r = 0, 1, \ldots \tag{54}$$

then

$$K = G \otimes \{E \oplus \rho_{n-1}\}. \tag{55}$$

Proof. Exactly as above, since ρ_{n-1} is improper. □

Note that, given the restrictions for n in (52) and (54) not all odd and even values of n are treated by theorems 10 and 11, respectively. The important dimensions 3 and 4 are missing but for them the methods of the next two sections may be used.

10. Double-group structure

Consider a set of operations $\{G\}$ from a Clifford algebra $\mathbb{C}^n$ that close if and only if the Pauli gauge is used (see (35)). Call E the identity, and $\widetilde{E}$ an operation such that

$$\widetilde{E} \notin \{G\}, \qquad \widetilde{E}^2 = E. \tag{56}$$

If E is taken as 1, then $\widetilde{E}$ may be taken as -1. We also require the condition

$$\widetilde{E}g = g\widetilde{E} = \tilde{g}, \qquad \forall g \in G. \tag{57}$$

The set $\{E \oplus \widetilde{E}\}$ forms a group K. Also, the set $\{G \oplus G\widetilde{E}\}$ that contains all the elements g and $\tilde{g}$ is now a group (in the Cartan gauge), because if $g^2 = -1$ as in (35), then g^2 equals $\widetilde{E}$ and the set closes. The group

$$\widetilde{G} = \{G \oplus G\widetilde{E}\} \tag{58}$$

is called the *double group* of G. Note that G is not a subgroup of $\widetilde{G}$ since it only closes under a multiplication rule (Pauli gauge) not valid in $\widetilde{G}$, for which, as for all our groups, the Cartan gauge must be used.

The relation between G, $K = E \oplus \widetilde{E}$ and $\widetilde{G}$ may be better understood by establishing a homomorphism of $\widetilde{G}$ onto K with G as its kernel (the set of $\widetilde{G}$ that maps onto the identity of K). Then $\widetilde{G}$ is an *extension* of K by G. (This is not a central extension, since G does not commute with all elements of $\widetilde{G}$.)

In the Clifford algebra, because of the commutation of $\widetilde{E}$, the double of an element such as $e_i e_j$ may be written indistinctly as $\tilde{e}_i e_j$ or $e_i \tilde{e}_j$. We shall use the following convention: the tilde shall always be applied to the first factor in a term. Thus, given

$$e_1 e_2 = -e_2 e_1 \quad \text{in } G, \tag{59}$$

we write

$$e_1 e_2 = \tilde{e}_2 e_1 \quad \text{in } \widetilde{G}. \tag{60}$$

If $\{G\} \in \mathbb{C}^n$ and n is even, ρ_n commutes with all terms of even rank (see (21)) and from (8), with $n = 4l$, it satisfies (57). Therefore, if $\{G\}$ contains only such terms of even rank, then

Table 1. Conjugation of $G_{\mathbf{w}}$.

Space $G_{\mathbf{w}}$	$\mathbf{x}$	$\mathbf{y}$	$\mathbf{z}$
Clifford elements	e_1	e_2	e_3
Rotation g_1	e_4	e_2	e_3
Rotation g_2	e_1	e_4	e_3
Rotation g_3	e_1	e_2	e_4

the reversion may be used for $\widetilde{E}$ in the above, so that the double group contains the reversion explicitly.

In $\mathbb{R}^3$ the double-group representations are associated with spin bases, but it should be understood that this is not so for the representations arising from the structure discussed in this section. Spin representations, however, do appear in $\mathbb{R}^4$ but, as we shall see later, have a different significance.

11. Generation of the proper octahedral group in $\mathbb{R}^4$ (group G_{192})

In $\mathbb{R}^3$, space spanned by $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, we have a proper group $\mathbf{O}$ of order 24, which we shall denote with G. In $\mathbb{R}^4$, space $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{w}$, $\mathbb{R}^3$ is orthogonal to $\mathbf{w}$ and we shall stress this by using the symbol $G_{\mathbf{w}}$ for G. We shall consider four $\mathbb{R}^3$ spaces embedded in $\mathbb{R}^4$ and their corresponding groups:

$$G_{\mathbf{w}}: \quad \mathbf{x}, \mathbf{y}, \mathbf{z}, \tag{61}$$

$$G_{\mathbf{x}}: \quad \mathbf{w}, \mathbf{y}, \mathbf{z}, \tag{62}$$

$$G_{\mathbf{y}}: \quad \mathbf{x}, \mathbf{w}, \mathbf{z}, \tag{63}$$

$$G_{\mathbf{z}}: \quad \mathbf{x}, \mathbf{y}, \mathbf{w}. \tag{64}$$

In order to generate $G_{\mathbf{x}}$ in (62) we have to rotate $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ into $\mathbf{w}$, $\mathbf{y}$, $\mathbf{z}$, for which we have to conjugate the original space, in principle, under a four-fold rotation (rotation by $\pi/2$). This operation is easy to obtain, by comparison with such operations in $G_{\mathbf{w}}$, although we shall find that the required operation is in fact an eight-fold rotation in $\mathbb{R}^4$. We shall now list the operation in question and its powers:

$$g_1 = \frac{1}{\sqrt{2}}(1 + e_4 e_1), \qquad g_1^2 = e_4 e_1, \qquad g_1^4 = -1, \qquad g_1^8 = 1. \tag{65}$$

It is easy to prove that $g_1^{-1} = \frac{1}{\sqrt{2}}(1 - e_4 e_1)$.

Similarly, we define

$$g_2 = \frac{1}{\sqrt{2}}(1 + e_4 e_2), \qquad g_3 = \frac{1}{\sqrt{2}}(1 + e_4 e_3). \tag{66}$$

It is now easy to perform rotations of e_1, e_2, e_3 by conjugating them with the above operations. The results are given in table 1.

Let us call G^4 the proper octahedral group in $\mathbb{R}^4$. We can expand it in coset representatives of its subgroup $G_{\mathbf{w}} \in \mathbb{R}^3$, and we shall take g_1, g_2, g_3 to be those coset representatives:

$$G^4 = G_{\mathbf{w}} \oplus g_1 G_{\mathbf{w}} \oplus g_2 G_{\mathbf{w}} \oplus g_3 G_{\mathbf{w}}. \tag{67}$$

We know that two such left cosets might be identical, say $g_1 G_{\mathbf{w}}$ and $g_2 G_{\mathbf{w}}$. Suppose that this is so. Then

$$G_{\mathbf{w}} = g_1^{-1} g_2 G_{\mathbf{w}}. \tag{68}$$

If this were so, because $E \in G_{\mathbf{w}}$ it must be identified with some operation on the right-hand side of (68), say

$$E = g_1^{-1} g_2 g, \qquad g \in G_{\mathbf{w}}. \tag{69}$$

But this is impossible since $g = g_2^{-1} g_1$ turns out to be

$$g = \tfrac{1}{2}(1 - e_4 e_2 + e_4 e_1 + e_2 e_1), \tag{70}$$

containing e_4, which does not exist in $G_{\mathbf{w}}$. Therefore, the four cosets are all distinct. Since the order of $G_{\mathbf{w}}$ is 24, G^4 is of order 96. Because the reversion $\rho_4 = e_1 e_2 e_3 e_4$ commutes with all operations in (70), and also $\rho_4^2 = 1$, it satisfies the same properties as $\widetilde{E}$ in (56) and (57) and can be used in it instead. This means that the group K defined in section 9 as $E \oplus \widetilde{E}$ in order to generate the double group can be taken to be

$$K = E \oplus \rho_4. \tag{71}$$

The corresponding double group $\widetilde{G}$,

$$\widetilde{G} = G^4 \oplus G^4 \rho_4, \tag{72}$$

is the cubic group $\mathbf{O}^4$ of order $96 \times 2 = 192$. Note that it contains eight-fold operations, as g_1 in (65).

12. From Clifford parameters to Cartesian matrices and vice versa

The procedure for deriving the Clifford parameters from the Cartesian matrices has been described by Pokorny, Herzig and Altmann [6].

For the inverse procedure the prescription is the following. Transform the Clifford units $e_1, \ldots, e_n$, which represent the reflections about the hyperplanes orthogonal to the basis vectors, by conjugation with the Clifford realization of the group operation g. According to Altmann [5] the conjugate $e_i^\dagger$ is

$$e_i^\dagger = g e_i g^{-1} \qquad \text{(proper operations)}, \tag{73}$$

$$e_i^\dagger = -g e_i g^{-1} \qquad \text{(improper operations)}. \tag{74}$$

Example 1. For the four-fold operation in $\mathbb{R}^4$, $g = \frac{1}{\sqrt{2}}(1 - e_4 e_1)$, given in (65),

$$e_1^\dagger = -e_4, \qquad e_2^\dagger = e_2, \qquad e_3^\dagger = e_3, \qquad e_4^\dagger = e_1. \tag{75}$$

This transformation can be written in the matrix form:

$$(e_1, e_2, e_3, e_4) \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} = (-e_4, e_2, e_3, e_1). \tag{76}$$

The matrix in (76), $\mathbf{A}$, transforms the (*polar*) vector $\mathbf{r}$ under the operation g (active picture):

$$g\mathbf{r} = \mathbf{A}\mathbf{r}. \tag{77}$$

Note that the components e_i of the row vectors in (76) are defined in function space while $\mathbf{r}$ in (77) is a vector in $\mathbb{R}^n$ and the matrix $\mathbf{A}$ is therefore on the left-hand side of $\mathbf{r}$.

13. Parametrization of rotations in $\mathbb{R}^4$ by the Clifford elements of pairs of rotations in $\mathbb{R}^3$

Such a parametrization is possible because the group of proper rotations in $\mathbb{R}^4$ can be considered as the direct product $SU(2) \otimes SU(2)$ [15].

Theorem 12. *A rotation in* $\mathbb{R}^4$ *in Clifford parametrization*

$$\mathsf{B} = b_0 1 + b_{32}e_3e_2 + b_{13}e_1e_3 + b_{21}e_2e_1 + b_{41}e_4e_1 + b_{42}e_4e_2 + b_{43}e_4e_3 + b_{1234}e_1e_2e_3e_4, \tag{78}$$

can be expressed by a pair of Clifford elements, where each of the elements describes a rotation in $\mathbb{R}^3$*:*

$$\mathsf{B} = ((b_0 + b_{1234}) + (b_{32} - b_{41})e_3e_2 + (b_{13} - b_{42})e_1e_3 + (b_{21} - b_{43})e_2e_1, \\ (b_0 - b_{1234}) + (b_{32} + b_{41})e_3e_2 + (b_{13} + b_{42})e_1e_3 + (b_{21} + b_{43})e_2e_1). \tag{79}$$

This parametrization will henceforth be referred to as direct-product parametrization.

Proof. As shown by Talman [15], an arbitrary rotation in $\mathbb{R}^4$ can always be written as a product of the following three factors: a rotation S in $\mathbb{R}^3$ (pole $\mathbf{s}$, angle σ), a rotation A (in $\mathbb{R}^4$) in the xy plane by an angle α combined with a rotation in the zw plane by $-\alpha$ and again a rotation T in $\mathbb{R}^3$ (pole $\mathbf{t}$, angle τ). □

In order to express a rotation in $\mathbb{R}^3$ by an angle ϕ about an axial unit vector $\mathbf{n}$ in terms of Clifford parameters, use (6) of [5]

$$\left[\left[\cos\frac{\phi}{2}, \mathbf{n}\sin\frac{\phi}{2}\right]\right] = \cos\frac{\phi}{2} + \sin\frac{\phi}{2}(n_x\boldsymbol{i} + n_y\boldsymbol{j} + n_z\boldsymbol{k}). \tag{80}$$

With the mapping (see (11) of [5])

$$e_3e_2 \leftrightarrow \boldsymbol{i}, \qquad e_1e_3 \leftrightarrow \boldsymbol{j}, \qquad e_2e_1 \leftrightarrow \boldsymbol{k}, \tag{81}$$

we obtain

$$\left[\left[\cos\frac{\phi}{2}, \mathbf{n}\sin\frac{\phi}{2}\right]\right] = \cos\frac{\phi}{2} + \sin\frac{\phi}{2}(n_xe_3e_2 + n_ye_1e_3 + n_ze_2e_1). \tag{82}$$

Similarly, a rotation by α in the xy plane is given by $\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}e_2e_1$ and a rotation by $-\alpha$ in the zw plane is given by $\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}e_4e_3$, the latter because e_4e_3 and e_2e_1 describe rotations in spaces orthogonal to each other. Therefore, the rotation (78) can be expressed as the product of the three Clifford elements, SAT, with

$$\mathsf{S} = \cos\frac{\sigma}{2} + \sin\frac{\sigma}{2}(s_xe_3e_2 + s_ye_1e_3 + s_ze_2e_1), \tag{83}$$

$$\mathsf{A} = \left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}e_2e_1\right)\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}e_4e_3\right), \tag{84}$$

$$\mathsf{T} = \cos\frac{\tau}{2} + \sin\frac{\tau}{2}(t_xe_3e_2 + t_ye_1e_3 + t_ze_2e_1). \tag{85}$$

Equating B and SAT, comparing the coefficients on both sides and forming the appropriate linear combinations of the b's, we obtain

$$b_0 - b_{1234} = \cos\frac{\sigma}{2}\cos\frac{\tau}{2} - \sin\frac{\sigma}{2}\sin\frac{\tau}{2}(\mathbf{s}\cdot\mathbf{t}) = \cos\frac{\delta}{2}, \tag{86}$$

$$b_{32} + b_{41} = \sin\frac{\sigma}{2}\cos\frac{\tau}{2}s_x + \cos\frac{\sigma}{2}\sin\frac{\tau}{2}t_x + \sin\frac{\sigma}{2}\sin\frac{\tau}{2}(\mathbf{s}\times\mathbf{t})_x = \sin\frac{\delta}{2}d_x, \tag{87}$$

$$b_{13} + b_{42} = \sin\frac{\sigma}{2}\cos\frac{\tau}{2}s_y + \cos\frac{\sigma}{2}\sin\frac{\tau}{2}t_y + \sin\frac{\sigma}{2}\sin\frac{\tau}{2}(\mathbf{s}\times\mathbf{t})_y = \sin\frac{\delta}{2}d_y, \tag{88}$$

$$b_{21} + b_{43} = \sin\frac{\sigma}{2}\cos\frac{\tau}{2}s_z + \cos\frac{\sigma}{2}\sin\frac{\tau}{2}t_z + \sin\frac{\sigma}{2}\sin\frac{\tau}{2}(\mathbf{s}\times\mathbf{t})_z = \sin\frac{\delta}{2}d_z, \tag{89}$$

where d_x, d_y, d_z are the components of the pole $\mathbf{d}$ and δ is the angle of a rotation (in $\mathbb{R}^3$) which is the combination of two rotations, namely one by an angle σ about $\mathbf{s}$ and the other by τ about $\mathbf{t}$ (see [16]). The part of (79) after the comma can therefore be written as

$$\boldsymbol{ST} = \cos\frac{\delta}{2} + \sin\frac{\delta}{2}(d_x e_3 e_2 + d_y e_1 e_3 + d_z e_2 e_1), \tag{90}$$

which is the product of the two Clifford elements:

$$\boldsymbol{S} = \cos\frac{\sigma}{2} + \sin\frac{\sigma}{2}(s_x e_3 e_2 + s_y e_1 e_3 + s_z e_2 e_1), \tag{91}$$

$$\boldsymbol{T} = \cos\frac{\tau}{2} + \sin\frac{\tau}{2}(t_x e_3 e_2 + t_y e_1 e_3 + t_z e_2 e_1). \tag{92}$$

In a similar fashion, the following linear combinations of b's are obtained (by tedious algebra):

$$\begin{aligned} b_0 + b_{1234} = {} & (b_0 - b_{1234})\cos\alpha \\ & + \left(-\sin\frac{\sigma}{2}\cos\frac{\tau}{2}s_z - \cos\frac{\sigma}{2}\sin\frac{\tau}{2}t_z + \sin\frac{\sigma}{2}\sin\frac{\tau}{2}[s_x t_y - s_y t_x]\right)\sin\alpha, \end{aligned} \tag{93}$$

$$\begin{aligned} b_{32} - b_{41} = {} & (b_{32} + b_{41})\cos\alpha \\ & + \left(\sin\frac{\sigma}{2}\cos\frac{\tau}{2}s_y - \cos\frac{\sigma}{2}\sin\frac{\tau}{2}t_y + \sin\frac{\sigma}{2}\sin\frac{\tau}{2}[-s_z t_x - s_x t_z]\right)\sin\alpha, \end{aligned} \tag{94}$$

$$\begin{aligned} b_{13} - b_{42} = {} & (b_{13} + b_{42})\cos\alpha \\ & + \left(-\sin\frac{\sigma}{2}\cos\frac{\tau}{2}s_x + \cos\frac{\sigma}{2}\sin\frac{\tau}{2}t_x + \sin\frac{\sigma}{2}\sin\frac{\tau}{2}[-s_y t_z - s_z t_y]\right)\sin\alpha, \end{aligned} \tag{95}$$

$$\begin{aligned} b_{21} - b_{43} = {} & (b_{21} + b_{43})\cos\alpha \\ & + \left(\cos\frac{\sigma}{2}\cos\frac{\tau}{2} + \sin\frac{\sigma}{2}\sin\frac{\tau}{2}[s_x t_x + s_y t_y - s_z t_z]\right)\sin\alpha. \end{aligned} \tag{96}$$

With

$$\boldsymbol{A} = \cos\alpha + \sin\alpha\, e_2 e_1 \tag{97}$$

the part before the comma in (79) can be shown to represent the product of the three Clifford elements $\boldsymbol{S}$, $\boldsymbol{A}$ and $\boldsymbol{T}$, which corresponds to a rotation in $\mathbb{R}^3$ by γ about a pole $\mathbf{c}$:

$$\boldsymbol{SAT} = \cos\frac{\gamma}{2}\boldsymbol{I} + \sin\frac{\gamma}{2}(c_x e_3 e_2 + c_y e_1 e_3 + c_z e_2 e_1). \tag{98}$$

Therefore, on introducing these results into (79) and on using the multiplication rules for products of direct products,

$$\mathsf{B} = (\boldsymbol{SAT}, \boldsymbol{ST}) = (\boldsymbol{S}, \boldsymbol{S})(\boldsymbol{A}, \boldsymbol{I})(\boldsymbol{T}, \boldsymbol{T}). \tag{99}$$

The three factors $(\boldsymbol{S}, \boldsymbol{S})$, $(\boldsymbol{A}, \boldsymbol{I})$ and $(\boldsymbol{T}, \boldsymbol{T})$ that we obtain in the direct-product parametrization (99) exactly correspond to the Clifford elements S, A and T in (83)–(85) thus completing the proof.

14. From the direct-product parameters to Cartesian matrices and vice versa

In comparison with Talman [15], whose arguments are based on $SU(2)$ matrices, we consider here Clifford elements, a strict homomorphism existing between the two concepts.

We start from a Clifford element $\boldsymbol{X}$ and regard x, y, z and w as coordinates of points on the unit sphere in four dimensions, provided that the condition $x^2 + y^2 + z^2 + w^2 = 1$ be imposed:

$$\boldsymbol{X} = w + xe_3e_2 + ye_1e_3 + ze_2e_1. \tag{100}$$

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two arbitrary Clifford elements corresponding to rotations in $\mathbb{R}^3$, then

$$\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X}\boldsymbol{B}^{-1} = w' + x'e_3e_2 + y'e_1e_3 + z'e_2e_1. \tag{101}$$

The numbers x', y', z', w', that appear in $\boldsymbol{X}'$, are linear functions of x, y, z, w and therefore the pair $(\boldsymbol{A}, \boldsymbol{B})$ generates a rotation of the unit sphere in $\mathbb{R}^4$ into itself; that is, a mapping has been established from the pair of rotations $(\boldsymbol{A}, \boldsymbol{B})$ in $\mathbb{R}^3$ onto a rotation in $\mathbb{R}^4$. The identity $\boldsymbol{A}(\boldsymbol{A}'\boldsymbol{X}\boldsymbol{B}'^{-1})\boldsymbol{B}^{-1} = (\boldsymbol{A}\boldsymbol{A}')\boldsymbol{X}(\boldsymbol{B}\boldsymbol{B}')^{-1}$ shows that the rotation generated by $(\boldsymbol{A}', \boldsymbol{B}')$ followed by the rotation generated by $(\boldsymbol{A}, \boldsymbol{B})$ is the same as the rotation generated by $(\boldsymbol{A}\boldsymbol{A}', \boldsymbol{B}\boldsymbol{B}')$, that is, that the mapping is homomorphic.

Example 2. Consider the operation given by $\left(\frac{1}{\sqrt{2}}[e_3e_2 + e_1e_3], \frac{1}{\sqrt{2}}[1 + e_2e_1]\right)$ which, as will be discussed in section 16, is shown to be an eight-fold rotation.

If $\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X}\boldsymbol{B}^{-1}$ is written in Clifford notation

$$\tfrac{1}{2}(e_3e_2 + e_1e_3)(w + xe_3e_2 + ye_1e_3 + ze_2e_1)(1 + e_2e_1) = -x + ze_3e_2 + we_1e_3 + ye_2e_1, \tag{102}$$

it is easy to deduce that

$$x' = z, \quad y' = w, \quad z' = y, \quad w' = -x, \tag{103}$$

which corresponds to the following Cartesian matrix (see (50) of [6]):

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}. \tag{104}$$

In order to proceed from a Cartesian matrix to the direct-product parameters, we start from $\boldsymbol{A}\boldsymbol{X} = \boldsymbol{X}'\boldsymbol{B}$, which written in component form reads

$$\begin{aligned}(a + a_xe_3e_2 + a_ye_1e_3 + a_ze_2e_1)&(w + xe_3e_2 + ye_1e_3 + ze_2e_1)\\ &= (w' + x'e_3e_2 + y'e_1e_3 + z'e_2e_1)(b + b_xe_3e_2 + b_ye_1e_3 + b_ze_2e_1).\end{aligned} \tag{105}$$

We multiply the matrices on both sides of (105) and define

$$\mathbf{A} = \begin{bmatrix} -a_x & -a_y & -a_z & a \\ a & -a_z & a_y & a_x \\ a_z & a & -a_x & a_y \\ -a_y & a_x & a & a_z \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} -b_x & -b_y & -b_z & b \\ b & b_z & -b_y & b_x \\ -b_z & b & b_x & b_y \\ b_y & -b_x & b & b_z \end{bmatrix}, \tag{106}$$

as well as the column vector

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}. \tag{107}$$

With an appropriate matrix **R**, (105) can be written as

$$\mathbf{Ar} = \mathbf{Br}' = \mathbf{BRr} \quad \Rightarrow \quad \mathbf{A} = \mathbf{BR}, \tag{108}$$

which allows the parameters $a, a_x, a_y, a_z, b, b_x, b_y, b_z$ to be calculated (up to a common phase factor).

Example 3. Consider the Cartesian matrix for the four-fold rotation given in (76). From the right-hand side of (108)

$$\begin{bmatrix} -a_x & -a_y & -a_z & a \\ a & -a_z & a_y & a_x \\ a_z & a & -a_x & a_y \\ -a_y & a_x & a & a_z \end{bmatrix} = \begin{bmatrix} -b & -b_y & -b_z & -b_x \\ -b_x & b_z & -b_y & b \\ -b_y & b & b_x & -b_z \\ -b_z & -b_x & b & b_y \end{bmatrix}, \tag{109}$$

the following conditions are obtained:

$$\begin{aligned} &a = b = a_x = -b_x \quad \Rightarrow \quad a = b = a_x = -b_x = \tfrac{1}{\sqrt{2}}, \\ &a_y = b_y = -b_y = a_z = b_z = -b_z \quad \Rightarrow \quad a_y = b_y = a_z = b_z = 0, \end{aligned} \tag{110}$$

and therefore

$$\boldsymbol{R} = \left(\tfrac{1}{\sqrt{2}}[1 + e_3 e_2], \tfrac{1}{\sqrt{2}}[1 - e_3 e_2]\right). \tag{111}$$

15. The different types of groups in $\mathbb{R}^4$

Four different cases can be distinguished depending on the structure of the groups in $\mathbb{R}^4$. These cases can be described according to section 9 by considering extensions of groups. In the following the direct-product notation is used, whereby $(\boldsymbol{g}_i, \boldsymbol{g}_j)$ denotes a rotation defined by the Clifford elements of the respective rotations in $\mathbb{R}^3$, g_i, g_j (section 12). When multiplying Clifford elements the appropriate projective factors, $[g_i, g_k]$, have to be taken into account:

$$g_i g_k = g_{ik} \quad \Rightarrow \quad \boldsymbol{g}_i \boldsymbol{g}_k = [g_i, g_k]\boldsymbol{g}_{ik}. \tag{112}$$

Case 1. For each group element $(\boldsymbol{g}_i, \boldsymbol{g}_j)$ the operation $(\boldsymbol{g}_i, -\boldsymbol{g}_j)$ does not belong to the group. Therefore, the operation $(E, -E)$, which is the reversion and is written as $e_1e_2e_3e_4$ in Clifford notation (see section 12), cannot be a group element. The *Pauli* gauge is used when multiplying the group elements:

$$(\boldsymbol{g}_i, \boldsymbol{g}_j)(\boldsymbol{g}_k, \boldsymbol{g}_l) = (\boldsymbol{g}_{ik}, \boldsymbol{g}_{jl}). \tag{113}$$

Case 2. For each group element $(\boldsymbol{g}_i, \boldsymbol{g}_j)$ the operation $(\boldsymbol{g}_i, -\boldsymbol{g}_j)$ belongs to the group, i.e., the reversion is a group element, and the *Pauli* gauge is used when multiplying the group elements:

$$(\boldsymbol{g}_i, \boldsymbol{g}_j)(\boldsymbol{g}_k, \boldsymbol{g}_l) = (\boldsymbol{g}_{ik}, \boldsymbol{g}_{jl}), \tag{114}$$

$$(\boldsymbol{g}_i, -\boldsymbol{g}_j)(\boldsymbol{g}_k, \boldsymbol{g}_l) = (\boldsymbol{g}_{ik}, -\boldsymbol{g}_{jl}). \tag{115}$$

Here the group can be considered as an extension of $\{E \oplus r\}$ by a group corresponding to case 1.

Case 3. For each group element $(\boldsymbol{g}_i, \boldsymbol{g}_j)$ the operation $(\boldsymbol{g}_i, -\boldsymbol{g}_j)$ does not belong to the group, i.e., the reversion is not a group element, but the *Cartan* gauge is used when multiplying the group elements:

$$(\boldsymbol{g}_i, \boldsymbol{g}_j)(\boldsymbol{g}_k, \boldsymbol{g}_l) = [g_i, g_k](\boldsymbol{g}_{ik}, \boldsymbol{g}_{jl}). \tag{116}$$

Table 2. A group of order 6 without reversion. Multiplication table obtained by assuming Pauli gauge.

	$(C^+_{3,10}, C^+_{32})$	$(C^-_{3,10}, C^-_{32})$	(C_{2c}, C_{2k})	(C_{2k}, C_{2g})	$(C_{2m}, \widetilde{C}_{2e})$
$(C^+_{3,10}, C^+_{32})$	$(C^-_{3,10}, C^-_{32})$	(E, E)	(C_{2k}, C_{2g})	$(C_{2m}, \widetilde{C}_{2e})$	(C_{2c}, C_{2k})
$(C^-_{3,10}, C^-_{32})$	(E, E)	$(C^+_{3,10}, C^+_{32})$	$(C_{2m}, \widetilde{C}_{2e})$	(C_{2c}, C_{2k})	(C_{2k}, C_{2g})
(C_{2c}, C_{2k})	$(C_{2m}, \widetilde{C}_{2e})$	(C_{2k}, C_{2g})	(E, E)	$(C^-_{3,10}, C^-_{32})$	$(C^+_{3,10}, C^+_{32})$
(C_{2k}, C_{2g})	(C_{2c}, C_{2k})	$(C_{2m}, \widetilde{C}_{2e})$	$(C^+_{3,10}, C^+_{32})$	(E, E)	$(C^-_{3,10}, C^-_{32})$
$(C_{2m}, \widetilde{C}_{2e})$	(C_{2k}, C_{2g})	(C_{2c}, C_{2k})	$(C^-_{3,10}, C^-_{32})$	$(C^+_{3,10}, C^+_{32})$	(E, E)

Table 3. A group of order 12 without reversion (see the text). Multiplication table obtained by assuming Cartan gauge.

	$(C^+_{3,10}, C^+_{32})$	$(C^-_{3,10}, C^-_{32})$	(C_{2c}, C_{2k})	(C_{2k}, C_{2g})	$(C_{2m}, \widetilde{C}_{2e})$
$(C^+_{3,10}, C^+_{32})$	$(\widetilde{C}^-_{3,10}, \widetilde{C}^-_{32})$	(E, E)	$(\widetilde{C}_{2k}, \widetilde{C}_{2g})$	$(C_{2m}, \widetilde{C}_{2e})$	(C_{2c}, C_{2k})
$(C^-_{3,10}, C^-_{32})$	(E, E)	$(\widetilde{C}^+_{3,10}, \widetilde{C}^+_{32})$	$(C_{2m}, \widetilde{C}_{2e})$	$(\widetilde{C}_{2c}, \widetilde{C}_{2k})$	(C_{2k}, C_{2g})
(C_{2c}, C_{2k})	$(C_{2m}, \widetilde{C}_{2e})$	$(\widetilde{C}_{2k}, \widetilde{C}_{2g})$	$(\widetilde{E}, \widetilde{E})$	$(C^-_{3,10}, C^-_{32})$	$(\widetilde{C}^+_{3,10}, \widetilde{C}^+_{32})$
(C_{2k}, C_{2g})	$(\widetilde{C}_{2c}, \widetilde{C}_{2k})$	$(C_{2m}, \widetilde{C}_{2e})$	$(C^+_{3,10}, C^+_{32})$	$(\widetilde{E}, \widetilde{E})$	$(\widetilde{C}^-_{3,10}, \widetilde{C}^-_{32})$
$(C_{2m}, \widetilde{C}_{2e})$	(C_{2k}, C_{2g})	(C_{2c}, C_{2k})	$(\widetilde{C}^-_{3,10}, \widetilde{C}^-_{32})$	$(\widetilde{C}^+_{3,10}, \widetilde{C}^+_{32})$	$(\widetilde{E}, \widetilde{E})$

Here $[g_i, g_k] = [g_j, g_l]$ must hold, because otherwise the reversion would appear as a group element contrary to the assumption, and the group can be considered as an extension of $\{E \oplus \widetilde{E}\}$ by a group corresponding to case 1.

Case 4. For each group element $(\boldsymbol{g}_i, \boldsymbol{g}_j)$ the operation $(\boldsymbol{g}_i, -\boldsymbol{g}_j)$ belongs to the group, i.e., the reversion is a group element, but the *Cartan* gauge is used when multiplying the group elements:

$$(\boldsymbol{g}_i, \boldsymbol{g}_j)(\boldsymbol{g}_k, \boldsymbol{g}_l) = [g_i, g_k](\boldsymbol{g}_{ik}, \boldsymbol{g}_{jl}), \tag{117}$$

$$(\boldsymbol{g}_i, -\boldsymbol{g}_j)(\boldsymbol{g}_k, \boldsymbol{g}_l) = [g_i, g_k](\boldsymbol{g}_{ik}, -\boldsymbol{g}_{jl}), \tag{118}$$

Here the group can be considered as an extension of $\{E \oplus r\}$ by a group corresponding to case 3, or as an extension of $\{E \oplus \widetilde{E}\}$ by a group of case 2, or as an extension of $\{E \oplus r \oplus \widetilde{E} \oplus \widetilde{E}r\}$ by a group of case 1.

Example 4. We consider a subgroup of the proper group of the pentatope (order 60). The parameters of the respective operations in $\mathbb{R}^3$ have been taken from pp 663 and 664 of [17]. If $(-\boldsymbol{g}_i, -\boldsymbol{g}_j)$ is considered to be equivalent to $(\boldsymbol{g}_i, \boldsymbol{g}_j)$ and $(-\boldsymbol{g}_i, \boldsymbol{g}_j)$ to be equivalent to $(\boldsymbol{g}_i, -\boldsymbol{g}_j)$ then the group of order 6, given in table 2, is obtained. Note that the tilde on top of C_{2e} is only necessary for compatibility with the definitions in table 75.1 of [17]. This group is an example for case 1.

If, however, the projective factors coming from the multiplication of the corresponding Clifford elements are retained, a group without reversion of order 12 is obtained which is the double group of the group of order 6 given in table 2. Although the full multiplication table is not provided in table 3, this can easily be generated by adding those (horizontal and vertical) entries which are created by multiplying the given ones by the reversion $(E, \widetilde{E})$. This group is an example for case 3.

In groups that contain the *reversion* (cases 2 and 4), for each element $(\boldsymbol{g}_i, \boldsymbol{g}_j)$ there exists an element $(\boldsymbol{g}_i, -\boldsymbol{g}_j)$ in the group. In Clifford notation the correspondence is as follows:

$$(\boldsymbol{g}_i, \boldsymbol{g}_j) \leftrightarrow b_0 + b_{32}e_3e_2 + b_{13}e_1e_3 + b_{21}e_2e_1 + b_{41}e_4e_1 + b_{42}e_4e_2 + b_{43}e_4e_3 + b_{1234}e_1e_2e_3e_4, \tag{119}$$

$$(\boldsymbol{g}_i, -\boldsymbol{g}_j) \leftrightarrow b_{1234} - b_{41}e_3e_2 - b_{42}e_1e_3 - b_{43}e_2e_1 - b_{32}e_4e_1 - b_{13}e_4e_2 - b_{21}e_4e_3 + b_{1234}e_1e_2e_3e_4. \tag{120}$$

Such groups have a formal double-group structure which has no relation to spin.

If a group in $\mathbb{R}^4$ contains the reversion $(E, \widetilde{E})$ as a group element (cases 2 and 4), then a theorem analogous to that formulated by Opechowski [18] for spin-double groups applies. The two different cases for the class structure depend on the distinction between *irregular* and *regular* operations (see [16]). The former (also called *bilateral binary* operations) are binary operations with a further binary operation perpendicular to them, the latter are all other operations.

Theorem 13

(i) If both g_i and g_j are regular in $\mathbb{R}^3$, then $(\boldsymbol{g}_i, \boldsymbol{g}_j)$ and $(\boldsymbol{g}_i, -\boldsymbol{g}_j)$ belong to two different classes in $\mathbb{R}^4$;

(ii) if either g_i or g_j, or both, are irregular in $\mathbb{R}^3$, then $(\boldsymbol{g}_i, \boldsymbol{g}_j)$ and $(\boldsymbol{g}_i, -\boldsymbol{g}_j)$ belong to the same class in $\mathbb{R}^4$.

16. What are the orders of the rotations in $\mathbb{R}^4$?

The order of an operation shall be the smallest power of the operation that equals the identity. The answer to this question is important if one constructs a proper point group in $\mathbb{R}^4$ by the direct-product parametrization using the operations of a group in $\mathbb{R}^3$. It is therefore interesting to know, when you start from an m-fold and an m'-fold rotation in two orthogonal two-dimensional subspaces, what will be the order of the resulting rotation in $\mathbb{R}^4$.

We start from the following two rotations:

$$g_1 \text{ is a rotation by an angle } \gamma = \frac{2k\pi}{m}, \qquad k < m;\, k, \quad m \in \mathbb{N}, \tag{121}$$

$$g_2 \text{ is a rotation by an angle } \delta = \frac{2k'\pi}{m'}, \qquad k' < 2m';\, k', \quad m' \in \mathbb{N}, \tag{122}$$

where k, m and k', m' are assumed to have no common divisor. Therefore, g_1 is an m-fold rotation and g_2 an m'-fold rotation. The angle γ is in the range $0 \leqslant \gamma < 2\pi$ while for δ the range can be doubled to become $0 \leqslant \delta < 4\pi$. This is so because when no reversion is involved the angle δ for g_2 is in the range $0 \leqslant \delta < 2\pi$, but when the reversion is present, operations $\tilde{g}_2$ must also be considered that correspond to a rotation angle in the range $2\pi \leqslant \delta < 4\pi$. The corresponding operation in $\mathbb{R}^4$ can either be written in terms of Clifford parameters:

$$\mathsf{B} = b_0 + b_{32}e_3e_2 + b_{13}e_1e_3 + b_{21}e_2e_1 + b_{41}e_4e_1 + b_{42}e_4e_2 + b_{43}e_4e_3 + b_{1234}e_1e_2e_3e_4, \tag{123}$$

or as

$$\mathsf{B} = (\boldsymbol{g}_1, \boldsymbol{g}_2), \tag{124}$$

where $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ are the Clifford elements for the rotations g_1 and g_2. From (86), (93) and (98) we have

$$\cos\frac{\gamma}{2} = b_0 + b_{1234}, \tag{125}$$

$$\cos\frac{\delta}{2} = b_0 - b_{1234}. \tag{126}$$

According to (53) of [6], (123) can be written as a product of two terms each of which describes a rotation in two orthogonal two-dimensional subspaces of $O(4)$, the rotation angles being ϕ_1 and ϕ_2, respectively:

$$\mathsf{B} = \left(\cos\frac{\phi_1}{2} + \mathsf{P}_1 \sin\frac{\phi_1}{2}\right)\left(\cos\frac{\phi_2}{2} + \mathsf{P}_2 \sin\frac{\phi_2}{2}\right). \tag{127}$$

For P_1 and P_2 the definitions of [6] apply. In general, their product must be

$$\mathsf{P}_1\mathsf{P}_2 = \pm e_1 e_2 e_3 e_4. \tag{128}$$

In the following we assume a minus sign in (128), but this has no influence on the final result.

Comparing the scalar part, i.e., the product of the two cosine terms in (127), with the scalar part in (123), we get:

$$\cos\frac{\phi_1}{2}\cos\frac{\phi_2}{2} = b_{1234}, \tag{129}$$

and similarly by comparing the coefficients of the reversion:

$$-\sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2} = b_0. \tag{130}$$

On the other hand, from (125) and (126) one obtains

$$b_0 = \frac{1}{2}\left(\cos\frac{\gamma}{2} + \cos\frac{\delta}{2}\right), \tag{131}$$

$$b_{1234} = \frac{1}{2}\left(\cos\frac{\gamma}{2} - \cos\frac{\delta}{2}\right). \tag{132}$$

Equating (129) and (131), and (130) and (132) it follows that

$$\phi_1 = \frac{\gamma+\delta}{2} \qquad \text{and} \qquad \phi_2 = \frac{\gamma-\delta}{2}, \tag{133}$$

where ϕ_2 may be zero (for $\gamma = \delta$).

For ϕ_1 and ϕ_2 (if not zero) the order of the corresponding rotations in two-dimensional subspaces can be calculated by forming the smallest multiples (n and n') of ϕ_1 and ϕ_2 divisible by 2π. The total order M for the rotations g_1, g_2 and for the resulting rotation in $\mathbb{R}^4$, N, is then the least common multiple of the respective partial orders (see table 4).

The general expression for the order of the resulting rotation in $\mathbb{R}^4$ is

$$N = \begin{cases} M & \text{if } M\frac{\gamma+\delta}{2\pi} \text{ even,} \\ 2M & \text{if } M\frac{\gamma+\delta}{2\pi} \text{ odd.} \end{cases} \tag{134}$$

Furthermore, *only for odd values of M multiplication of an operation by the reversion can change the order N*. This can easily be seen, because the addition of 2π to δ changes $M\frac{\gamma+\delta}{2\pi}$ from even to odd or *vice versa* only if M is odd, but not if M is even. In the latter case $M\frac{\gamma+\delta}{2\pi}$ remains either even or odd by adding 2π to δ.

For rotations in $\mathbb{R}^4$ that are compatible with crystallographic symmetry the corresponding Hermann [19] symbol is given in the last column of table 4. Non-crystallographic operations are marked with ‘n.c.’ instead.

Table 4. Examples for obtaining the orders of rotations in $\mathbb{R}^4$.

(g_1, g_2)	γ	k	m	δ	k'	m'	M	ϕ_1	n	ϕ_2	n'	N	
(E, E)	0	0	1	0	0	1	1	0	1	0	1	1	1111
(C_2, E)	π	1	2	0	0	1	2	$\frac{\pi}{2}$	4	$\frac{\pi}{2}$	4	4	44
(C_3, E)	$\frac{2\pi}{3}$	1	3	0	0	1	3	$\frac{\pi}{3}$	6	$\frac{\pi}{3}$	6	6	66
(C_4, E)	$\frac{\pi}{2}$	1	4	0	0	1	4	$\frac{\pi}{4}$	8	$\frac{\pi}{4}$	8	8	n.c.
(C_5, E)	$\frac{2\pi}{5}$	1	5	0	0	1	5	$\frac{\pi}{5}$	10	$\frac{\pi}{5}$	10	10	n.c.
(C_6, E)	$\frac{\pi}{3}$	1	6	0	0	1	6	$\frac{\pi}{6}$	12	$\frac{\pi}{6}$	12	12	n.c.
$(E, \widetilde{E})$	0	0	1	2π	1	1	1	π	2	π	2	2	2222
$(C_2, \widetilde{E})$	π	1	2	2π	1	1	2	$\frac{3\pi}{2}$	4	$\frac{\pi}{2}$	4	4	44
$(C_3, \widetilde{E})$	$\frac{2\pi}{3}$	1	3	2π	1	1	3	$\frac{4\pi}{3}$	3	$\frac{2\pi}{3}$	3	3	33
$(C_4, \widetilde{E})$	$\frac{\pi}{2}$	1	4	2π	1	1	4	$\frac{5\pi}{4}$	8	$\frac{3\pi}{4}$	8	8	n.c.
$(C_5, \widetilde{E})$	$\frac{2\pi}{5}$	1	5	2π	1	1	5	$\frac{6\pi}{5}$	5	$\frac{4\pi}{5}$	5	5	n.c.
$(C_6, \widetilde{E})$	$\frac{\pi}{3}$	1	6	2π	1	1	6	$\frac{7\pi}{6}$	12	$\frac{5\pi}{6}$	12	12	n.c.
(C_3, C_3)	$\frac{2\pi}{3}$	1	3	$\frac{2\pi}{3}$	1	3	3	$\frac{2\pi}{3}$	3	0	1	3	311
(C_4, C_4)	$\frac{\pi}{2}$	1	4	$\frac{\pi}{2}$	1	4	4	$\frac{\pi}{2}$	4	0	1	4	411
(C_5, C_5)	$\frac{2\pi}{5}$	1	5	$\frac{2\pi}{5}$	1	5	5	$\frac{2\pi}{5}$	5	0	1	5	n.c.
(C_6, C_6)	$\frac{\pi}{3}$	1	6	$\frac{\pi}{3}$	1	6	6	$\frac{\pi}{3}$	6	0	1	6	611
$(C_2, \widetilde{C}_2)$	π	1	2	3π	3	2	2	2π	1	π	2	2	2211
$(C_3, \widetilde{C}_3)$	$\frac{2\pi}{3}$	1	3	$\frac{8\pi}{3}$	4	3	3	$\frac{5\pi}{3}$	6	π	2	6	622
$(C_4, \widetilde{C}_4)$	$\frac{\pi}{2}$	1	4	$\frac{5\pi}{2}$	5	4	4	$\frac{3\pi}{2}$	4	π	2	4	422
$(C_5, \widetilde{C}_5)$	$\frac{2\pi}{5}$	1	5	$\frac{12\pi}{5}$	6	5	5	$\frac{7\pi}{5}$	10	π	2	10	n.c.
$(C_6, \widetilde{C}_6)$	$\frac{\pi}{3}$	1	6	$\frac{7\pi}{3}$	7	6	6	$\frac{4\pi}{3}$	3	π	2	6	322
(C_2, C_3)	π	1	2	$\frac{2\pi}{3}$	1	3	6	$\frac{5\pi}{6}$	12	$\frac{\pi}{6}$	12	12	T
(C_2, C_4)	π	1	2	$\frac{\pi}{2}$	1	4	4	$\frac{3\pi}{4}$	8	$\frac{\pi}{4}$	8	8	8
(C_2, C_5)	π	1	2	$\frac{2\pi}{5}$	1	5	10	$\frac{7\pi}{10}$	20	$\frac{3\pi}{10}$	20	20	n.c.
(C_2, C_6)	π	1	2	$\frac{\pi}{3}$	1	6	6	$\frac{2\pi}{3}$	3	$\frac{\pi}{3}$	6	6	63
(C_{12}^5, C_{12})	$\frac{5\pi}{6}$	5	12	$\frac{\pi}{6}$	1	12	12	$\frac{\pi}{2}$	4	$\frac{\pi}{3}$	6	12	64
$(C_{12}^5, \widetilde{C}_{12})$	$\frac{5\pi}{6}$	5	12	$\frac{13\pi}{6}$	13	12	12	$\frac{3\pi}{2}$	4	$\frac{2\pi}{3}$	3	12	43
(C_5^2, C_5)	$\frac{4\pi}{5}$	2	5	$\frac{2\pi}{5}$	1	5	5	$\frac{3\pi}{5}$	10	$\frac{\pi}{5}$	10	10	X
$(C_5^2, \widetilde{C}_5)$	$\frac{4\pi}{5}$	2	5	$\frac{12\pi}{5}$	6	5	5	$\frac{8\pi}{5}$	5	$\frac{4\pi}{5}$	5	5	5

17. Poles of proper operations in $\mathbb{R}^4$

Example 5. Consider the eight-fold rotation given in [6],

$$g = \frac{1}{\sqrt{8}}(1 + e_3e_2 + e_1e_3 + e_2e_1 + e_4e_1 + e_4e_2 - e_4e_3 - e_1e_2e_3e_4). \tag{135}$$

In terms of direct-product parameters we have

$$g = \left(\frac{1}{\sqrt{2}}[e_3e_2 + e_1e_3], \frac{1}{\sqrt{2}}[1 + e_2e_1]\right). \tag{136}$$

The pole, Π_g, of the $\mathbb{R}^4$ operation g is obtained by replacing in (136) the Clifford elements for the two rotations in $\mathbb{R}^3$ by the corresponding expressions of their poles

$$\Pi_g = \left(\frac{1}{\sqrt{2}}[e_3e_2 + e_1e_3], e_2e_1\right). \tag{137}$$

Table 5. Clifford parameters for the cyclic group of order 8 in $\mathbb{R}^4$.

E	1
C_8^+	$\frac{1}{\sqrt{8}}(1 + e_3e_2 + e_1e_3 + e_2e_1 - e_4e_1 - e_4e_2 + e_4e_3 - e_1e_2e_3e_4)$
C_4^+	$\frac{1}{2}(-1 + e_2e_1 + e_4e_3 - e_1e_2e_3e_4)$
C_8^{3+}	$\frac{1}{\sqrt{8}}(-1 - e_3e_2 - e_1e_3 + e_2e_1 + e_4e_1 + e_4e_2 + e_4e_3 + e_1e_2e_3e_4)$
r	$e_1e_2e_3e_4$
C_8^{3-}	$\frac{1}{\sqrt{8}}(-1 + e_3e_2 + e_1e_3 - e_2e_1 - e_4e_1 - e_4e_2 - e_4e_3 + e_1e_2e_3e_4)$
C_4^-	$\frac{1}{2}(-1 - e_2e_1 - e_4e_3 - e_1e_2e_3e_4)$
C_8^-	$\frac{1}{\sqrt{8}}(1 - e_3e_2 - e_1e_3 - e_2e_1 + e_4e_1 + e_4e_2 - e_4e_3 - e_1e_2e_3e_4)$

Table 6. Poles $\{\mathbf{n}_1, \mathbf{n}_2\}$ for the operations of the cyclic group of order 8 in $\mathbb{R}^4$.

	$\mathbf{n}_1$	$\mathbf{n}_2$
E	$(0\,0\,0)$	$(0\,0\,0)$
C_8^+	$\left(\frac{1}{\sqrt{2}}\,\frac{1}{\sqrt{2}}\,0\right)$	$(0\,0\,1)$
C_4^+	$(0\,0\,0)$	$(0\,0\,1)$
C_8^{3+}	$\left(-\frac{1}{\sqrt{2}}\,-\frac{1}{\sqrt{2}}\,0\right)$	$(0\,0\,1)$
r	$(0\,0\,0)$	$(0\,0\,0)$
C_8^{3-}	$\left(\frac{1}{\sqrt{2}}\,\frac{1}{\sqrt{2}}\,0\right)$	$(0\,0\,-1)$
C_4^-	$(0\,0\,0)$	$(0\,0\,-1)$
C_8^-	$\left(-\frac{1}{\sqrt{2}}\,-\frac{1}{\sqrt{2}}\,0\right)$	$(0\,0\,-1)$

The pole Π_g may therefore be defined as the set of two poles for the rotations in $\mathbb{R}^3$ given on the right-hand side of (136):

$$\Pi_g = \{\mathbf{n}_1, \mathbf{n}_2\}, \qquad \mathbf{n}_1 = \left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right), \qquad \mathbf{n}_2 = (0, 0, 1). \tag{138}$$

Alternatively, the pole can be written as a single Clifford element by performing the appropriate transformation of (137):

$$\Pi_g = \tfrac{1}{\sqrt{8}}(e_3e_2 + e_1e_3 + \sqrt{2}e_2e_1 - e_4e_1 - e_4e_2 + \sqrt{2}e_4e_3). \tag{139}$$

The last expression agrees with the second pole of [6] (the other pole is orthogonal to that in (139) and provides no further information).

Example 6. Consider the cyclic group of order 8 in O(4).

The Clifford parameters of the operations are given in table 5. If the poles of the operations of the cyclic group of order 8 in $\mathbb{R}^4$ (table 6) are defined in terms of Clifford, they are all different. However, this is not so for the pairs of vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ listed in table 6.

If $\{\mathbf{n}_1, \mathbf{n}_2\}$ is the pole of a rotation, then the antipole is $\{-\mathbf{n}_1, -\mathbf{n}_2\}$; $\{\mathbf{n}_1, -\mathbf{n}_2 t\}$ is the pole of the rotation multiplied by the reversion r, and $\{-\mathbf{n}_1, \mathbf{n}_2\}$ is the antipole of the rotation multiplied by the reversion. In table 6 the symbol $(0\,0\,0)$ denotes an arbitrary vector. In this sense the poles $\{\mathbf{n}_1, \mathbf{n}_2\}$ for C_8^+ and C_4^+, e.g., can be considered as identical.

18. Poles of improper operations in $\mathbb{R}^4$

In general, improper operations may be considered as the product of a reflection and a rotation. An improper operation in $\mathbb{R}^4$ can always be split into a reflection and a rotation confined to a two-dimensional subspace. (In terms of Clifford parameters such a rotation is one for which the reversion does not appear explicitly.)

In order to show this we start from the general expression for an improper operation in terms of Clifford:

$$g = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_{234}e_2e_3e_4 + a_{134}e_1e_3e_4 + a_{124}e_1e_2e_4 + a_{123}e_1e_2e_3. \tag{140}$$

We now define the norm for the first four terms (which describe a pure reflection):

$$a = \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2}, \tag{141}$$

and, similarly, the norm for the last four terms:

$$\bar{a} = \sqrt{a_{234}^2 + a_{134}^2 + a_{124}^2 + a_{123}^2}. \tag{142}$$

Taking into account that every reflection is identical to its own inverse, we obtain from (140):

$$g = \frac{1}{a}(a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4)\left\{a + \frac{1}{a}(a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4) \right.$$
$$\left. \times (a_{234}e_2e_3e_4 + a_{134}e_1e_3e_4 + a_{124}e_1e_2e_4 + a_{123}e_1e_2e_3)\right\}. \tag{143}$$

Now the expression inside the curly brackets in (143) is multiplied and the following condition is taken into consideration:

$$a_4a_{123} - a_1a_{234} + a_2a_{134} - a_3a_{124} = 0, \tag{144}$$

which comes from the requirement that the square of the normalized Clifford element (140) must also be normalized. This leads to the following result:

$$g = \frac{1}{a}(a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4) \times \frac{1}{a}\{a^2 + (-a_1a_{123} - a_4a_{234})e_3e_2$$
$$+ (-a_2a_{123} + a_4a_{134})e_1e_3 + (-a_3a_{123} - a_4a_{124})e_2e_1 + (a_2a_{124} + a_3a_{134})e_4e_1$$
$$+ (-a_1a_{124} + a_3a_{234})e_4e_2 + (-a_1a_{134} - a_2a_{234})e_4e_3\}, \tag{145}$$

which shows that an arbitrary improper operation can be expressed as the product of a pure reflection g_1 and a rotation in a two-dimensional subspace g_2, because owing to condition (144) the coefficient of the reversion $(e_1e_2e_3e_4)$ is zero in (145).

The pole of the reflection g_1 is identical to the operation itself

$$\Pi_{g_1} = \frac{1}{a}(a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4), \tag{146}$$

while for a rotation in a two-dimensional subspace the pole is obtained by omitting the scalar part of the Clifford element and performing renormalization

$$\Pi_{g_2} = \frac{1}{\bar{a}}\{(-a_1a_{123} - a_4a_{234})e_3e_2 + (-a_2a_{123} + a_4a_{134})e_1e_3 + (-a_3a_{123} - a_4a_{124})e_2e_1$$
$$+ (a_2a_{124} + a_3a_{134})e_4e_1 + (-a_1a_{124} + a_3a_{234})e_4e_2 + (-a_1a_{134} - a_2a_{234})e_4e_3\}. \tag{147}$$

The pole for the total operation $g = g_1g_2$ is obtained as product of Π_{g_1} and Π_{g_2}:

$$\Pi_g = \Pi_{g_1}\Pi_{g_2} = \Pi_{g_2}\Pi_{g_1}, \tag{148}$$

where Π_{g_1} and Π_{g_2} can be shown to commute, whence

$$\Pi_g = \frac{1}{\bar{a}}(a_{234}e_2e_3e_4 + a_{134}e_1e_3e_4 + a_{124}e_1e_2e_4 + a_{123}e_1e_2e_3). \tag{149}$$

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